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2007 J. Phys. A: Math. Theor. 40 F113

(http://iopscience.iop.org/1751-8121/40/4/F01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) F113–F117 doi:10.1088/1751-8113/40/4/F01

FAST TRACK COMMUNICATION

On the number of bound states for weak perturbations of spin–orbit Hamiltonians

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Received 29 November 2006, in final form 11 December 2006
Published 9 January 2007
Online at stacks.iop.org/JPhysA/40/F113

Abstract
We give a variational proof of the existence of infinitely many bound states below the continuous spectrum for some weak perturbations of a class of spin–orbit Hamiltonians including the Rashba and Dresselhaus Hamiltonians.

PACS numbers: 71.70.Ej, 73.63.Hs

In the recent paper [3] Chaplik and Magarill have discovered a surprising fact: the Rashba Hamiltonian H_R,

$$H_R = \begin{pmatrix} p^2 & \alpha_R(p_y + \mathrm{i}p_x) \\ \alpha_R(p_y - \mathrm{i}p_x) & p^2 \end{pmatrix}$$

(the real parameter α_R is the Rashba constant expressing the strength of the spin–orbit coupling [2, 7]) perturbed by a short-range rotationally symmetric negative potential has an infinite number of eigenvalues below the threshold of the continuous spectrum. More precisely, for a rotationally symmetric shallow potential well V with $mUR^2/\hbar^2 \ll 1$, where m is the effective mass and U and R are the depth and radius of the well, respectively, in [3] a system of equations was derived to be satisfied by the eigenvalues, and it was shown (using the pole approximation for the calculation of some integrals) that the system has an infinite number of solutions below the continuous spectrum of $H_R + V$.

In the present communication we are going to provide a strict mathematical justification for the existence of infinitely many bound states below the continuous spectrum for short-range perturbations of a much larger class of spin–orbit Hamiltonians, which includes, in particular, the above Rashba Hamiltonian as well as the Dresselhaus Hamiltonian [7],

$$H_D = \begin{pmatrix} p^2 & -\alpha_D(p_x + \mathrm{i}p_y) \\ -\alpha_D(p_x - \mathrm{i}p_y) & p^2 \end{pmatrix}$$

(here α_D, a real parameter, is the Dresselhaus constant). Our technique is elementary and uses the max–min principle in the spirit of [8]. Moreover, the potential well V can be non-symmetric, and 'shallow' in our context means $V \in L^1$.

We denote by $\mathcal{H}$ the Hilbert space $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ of two-dimensional spinors; by $\mathcal{F}$ we denote the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$; then $\mathcal{F}_2 := \mathcal{F} \otimes 1_{\mathbb{C}^2}$ is the Fourier transform in $\mathcal{H}$. Let H_0 be a self-adjoint operator in $\mathcal{H}$ whose Fourier transform $\widehat{H}_0 := \mathcal{F}_2 H_0 \mathcal{F}_2^{-1}$ is the multiplication by the matrix

$$\widehat{H}_0(\mathbf{p}) = \begin{pmatrix} p^2 & A(\mathbf{p}) \\ A^*(\mathbf{p}) & p^2 \end{pmatrix}, \qquad \mathbf{p} \in \mathbb{R}^2, \tag{1}$$

where A is a continuous complex function on $\mathbb{R}^2$, star (*) means the complex conjugation and, as usual, $p := |\mathbf{p}|$. Obviously, H_0 is self-adjoint. The Rashba and Dresselhaus Hamiltonians above have the form (1) with a linear A. In generalizing the linearity we assume

$$\limsup_{p\to\infty} \frac{|A(\mathbf{p})|}{p^2} < 1. \tag{2}$$

Clearly, H_0 has no discrete spectrum; its spectrum is the union of the ranges of two functions $\lambda_\pm$ (dispersion laws): $\lambda_\pm(\mathbf{p}) = p^2 \pm |A(\mathbf{p})|$, hence $\operatorname{spec} H_0 = [\kappa, +\infty)$, where $\kappa := \inf\{p^2 - |A(\mathbf{p})| : \mathbf{p} \in \mathbb{R}^2\} > -\infty$. Moreover, there is a unitary matrix $M(\mathbf{p})$ depending continuously on $\mathbf{p} \in \mathbb{R}^2$ such that

$$M(\mathbf{p})\widehat{H}_0(\mathbf{p})M^*(\mathbf{p}) = \begin{pmatrix} \lambda_+(\mathbf{p}) & 0 \\ 0 & \lambda_-(\mathbf{p}) \end{pmatrix}, \qquad \mathbf{p} \in \mathbb{R}^2. \tag{3}$$

Denote $S := \{\mathbf{p} \in \mathbb{R}^2 : \lambda_-(\mathbf{p}) = \kappa\}$; this is a non-empty compact set. We will assume that

$$\text{the function } |A(\mathbf{p})| \text{ is of class } C^2 \text{ in a neighbourhood of } S. \tag{4}$$

For the Rashba and Dresselhaus Hamiltonians one has $\kappa = -\alpha_J^2/4$ $(J = R, D)$ and S is the circle $\{\mathbf{p} : 2p = |\alpha_J|\}$; in these cases S is called *the loop of extrema*. The condition (4) is obviously satisfied for these Hamiltonians.

The two conditions (2) and (4) imply that for every $\mathbf{p}_0 \in S$ there is a constant $c > 0$ such that we have

$$0 \leqslant \lambda_-(\mathbf{p}) - \kappa \leqslant c(\mathbf{p} - \mathbf{p}_0)^2 \tag{5}$$

for all $\mathbf{p} \in \mathbb{R}^2$.

Now let V be a real-valued scalar potential from $L^p(\mathbb{R}^2)$ with some $p > 1$. Using the Sobolev inequality and an explicit form for the Green function of $-\Delta$ we see that $V(-\Delta + E)^{-1}$ with $E > 0$ is a Hilbert–Schmidt operator; therefore, V is a compact perturbation of $(-\Delta) \oplus (-\Delta)$ (we denote $V \oplus V = V I_{\mathbb{C}^2}$, where $I_{\mathbb{C}^2}$ is the identity operator in $\mathbb{C}^2$, by the symbol V since this notation does not lead to confusion). Using (2) it is easy to show that the domains of $(-\Delta) \oplus (-\Delta)$ and H_0 coincide and the graph norms in these domains are equivalent. Hence, V is a relatively compact perturbation of H_0. As a result, we get that the operator $H := H_0 + V$ is well defined and that $\operatorname{spec}_{\mathrm{ess}} H = [\kappa, +\infty)$.

Below, for a distribution f by $\widehat{f}$ we denote its Fourier transform. To avoid mixing terminology, an Hermitian $n \times n$ matrix C will be called *positive definite* if $\langle \xi | C\xi \rangle > 0$ for any non-zero $\xi \in \mathbb{C}^n$, and will be called *positive semi-definite* if the above equality is non-strict. By analogy one introduces *negative definite* and *negative semi-definite* matrices.

Theorem 1. *Let $N \in \mathbb{N}$. Assume that $V \in L^1(\mathbb{R}^2)$ and that $\widehat{V}$ satisfies the following condition: there are N points $\mathbf{p}_1, \ldots, \mathbf{p}_N \in S$ such that the matrix $(\widehat{V}(\mathbf{p}_m - \mathbf{p}_n))_{1\leqslant m,n\leqslant N}$ is negative definite. Then H has at least N eigenvalues, counting multiplicity, below κ.*

Proof. According to the max–min principle, it is sufficient to show that we can find N vectors $\Psi_m \in \mathcal{H}$, $m = 1, \ldots, N$, such that the matrix with the entries $\langle \Psi_m | (H-\kappa)\Psi_n\rangle$, $1 \leqslant m, n \leqslant N$, is negative definite; the vectors Ψ_m are then *a posteriori* linearly independent.

Denote $f_a(\mathbf{x}) := \exp\left(-\frac{1}{2}|\mathbf{x}|^a\right)$, $\mathbf{x} \in \mathbb{R}^2$, with $a > 0$. As observed in [8],

$$\int_{\mathbb{R}^2} |\nabla f_a(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} = \frac{\pi}{2} a. \tag{6}$$

Furthermore, by the Lebesgue dominated convergence theorem,

$$\lim_{a\to 0} \int_{\mathbb{R}^2} V(\mathbf{x}) |f_a(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} = \mathrm{e}^{-1} \int_{\mathbb{R}^2} V(\mathbf{x}) \, \mathrm{d}\mathbf{x}. \tag{7}$$

Let $\widehat{f}_a$ be the Fourier transform of f_a. Take spinors Ψ_m such that their Fourier transforms $\widehat{\Psi}_m$ are of the form $\widehat{\Psi}_m(\mathbf{p}) = M(\mathbf{p})\psi_m(\mathbf{p})$, where

$$\psi_m(\mathbf{p}) = \begin{pmatrix} 0 \\ \widehat{f}_a(\mathbf{p} - \mathbf{p}_m) \end{pmatrix} \tag{8}$$

and $M(\mathbf{p})$ is taken from (3). We show that if a is sufficiently small, then the matrix $(\langle \Psi_m | (H - \kappa)\Psi_n\rangle)$ is negative definite. For this purpose it is sufficient to show that

$$\lim_{a\to 0} \langle \Psi_m | (H_0 - \kappa)\Psi_n\rangle = 0, \tag{9}$$

$$\lim_{a\to 0} \langle \Psi_m | V\Psi_n\rangle = 2\pi \, \mathrm{e}^{-1} \widehat{V}(\mathbf{p}_m - \mathbf{p}_n) \tag{10}$$

for all (m, n).

By definition of Ψ_m one has

$$|\langle \Psi_m | (H_0 - \kappa)\Psi_n\rangle| = \left| \int_{\mathbb{R}^2} (\lambda_-(\mathbf{p}) - \kappa) \widehat{f}_a^*(\mathbf{p} - \mathbf{p}_m) \widehat{f}_a(\mathbf{p} - \mathbf{p}_n) \, \mathrm{d}\mathbf{p} \right|$$
$$\leqslant \left[\int_{\mathbb{R}^2} (\lambda_-(\mathbf{p}) - \kappa) |\widehat{f}_a(\mathbf{p} - \mathbf{p}_m)|^2 \, \mathrm{d}\mathbf{p} \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^2} (\lambda_-(\mathbf{p}) - \kappa) |\widehat{f}_a(\mathbf{p} - \mathbf{p}_n)|^2 \, \mathrm{d}\mathbf{p} \right]^{\frac{1}{2}}.$$

On the other hand, by (5) and (6) one has

$$0 \leqslant \int_{\mathbb{R}^2} (\lambda_-(\mathbf{p}) - \kappa) |\widehat{f}_a(\mathbf{p} - \mathbf{p}_m)|^2 \, \mathrm{d}\mathbf{p}$$
$$\leqslant c \int_{\mathbb{R}^2} (\mathbf{p} - \mathbf{p}_m)^2 |\widehat{f}_a(\mathbf{p} - \mathbf{p}_m)|^2 \, \mathrm{d}\mathbf{p} = c \int_{\mathbb{R}^2} \mathbf{p}^2 |\widehat{f}_a(\mathbf{p})|^2 \, \mathrm{d}\mathbf{p} = \frac{\pi}{2} ca,$$

which proves (9). As for (10), one has

$$\langle \Psi_m | V\Psi_n\rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle \widehat{\Psi}_m(\mathbf{p}) | \widehat{V}(\mathbf{p} - \mathbf{q}) \widehat{\Psi}_n(\mathbf{q})\rangle \, \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{q}$$
$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle \psi_m(\mathbf{p}) | \widehat{V}(\mathbf{p} - \mathbf{q}) \psi_n(\mathbf{q})\rangle \, \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{q}$$

since the matrices $\widehat{V}(\mathbf{p}-\mathbf{q})$ and $M(\mathbf{p})$ commute. On the other hand,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle\psi_m(\mathbf{p})|\widehat{V}(\mathbf{p}-\mathbf{q})\psi_n(\mathbf{q})\rangle\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widehat{V}(\mathbf{p}-\mathbf{q})\widehat{f}_a^*(\mathbf{p}-\mathbf{p}_m)\widehat{f}_a(\mathbf{q}-\mathbf{p}_n)\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}$$

$$=\int_{\mathbb{R}^2}V(\mathbf{x})\,\mathrm{e}^{\mathrm{i}(\mathbf{p}_m-\mathbf{p}_n)\mathbf{x}}\left|f_a(\mathbf{x})\right|^2\mathrm{d}\mathbf{x}\xrightarrow{a\to 0}2\pi\,\mathrm{e}^{-1}\widehat{V}(\mathbf{p}_m-\mathbf{p}_n).$$

The proof is complete. □

Let us list several corollaries.

Corollary 2. *If $\int_{\mathbb{R}^2}V(\mathbf{x})\,\mathrm{d}\mathbf{x}<0$, then H has at least one eigenvalue below κ.*

Proof. Since S is non-empty, it remains to note that $\widehat{V}(\mathbf{0})\equiv\frac{1}{2\pi}\int_{\mathbb{R}^2}V(\mathbf{x})\,\mathrm{d}\mathbf{x}$. □

Note that taking $A=0$ we recover a result of Simon: a weak negative perturbation of the free Hamiltonian $-\Delta$ in two dimensions always has a bound state below the threshold of the continuous spectrum [6].

Below by $\#S$ we denote the number of points in S, if S is finite, and ∞, otherwise.

Corollary 3. *Let V be non-positive and non-vanishing on a set of positive measure. Then H has at least $\#S$ eigenvalues below κ counting multiplicities.*

Proof. It is sufficient to show that the matrix $(\widehat{V}(\mathbf{p}_m-\mathbf{p}_n))_{1\leqslant m,n\leqslant N}$ is negative definite for every choice of points $\mathbf{p}_1,\ldots,\mathbf{p}_N\in\mathbb{R}^2$. By the Bochner theorem, $-\sum_{mn}\widehat{V}(\mathbf{p}_m-\mathbf{p}_n)\xi_m^*\xi_n\geqslant 0$ for any $(\xi_m)\in\mathbb{C}^N$ and it remains to note that $\sum_{mn}\widehat{V}(\mathbf{p}_m-\mathbf{p}_n)\xi_m^*\xi_n\neq 0$ for $(\xi_m)\neq 0$. In fact, if $\sum_{mn}\widehat{V}(\mathbf{p}_m-\mathbf{p}_n)\xi_m^*\xi_n=0$, then

$$\int_{\mathbb{R}^2}\left|\sum_m\xi_m\,\mathrm{e}^{\mathrm{i}\mathbf{p}_m\mathbf{x}}\right|^2V(\mathbf{x})\,\mathrm{d}\mathbf{x}=0;$$

therefore, $\sum_m\xi_m\mathrm{e}^{\mathrm{i}\mathbf{p}_m\mathbf{x}}=0$ on the support of V. Since exponents $\mathrm{e}^{\mathrm{i}\mathbf{p}_m\mathbf{x}}$ are real-analytic in $\mathbf{x}$ and $\sum_m\xi_m\mathrm{e}^{\mathrm{i}\mathbf{p}_m\mathbf{x}}=0$ on a set of non-zero Lebesgue measure, the equality $\sum_m\xi_m\mathrm{e}^{\mathrm{i}\mathbf{p}_m\mathbf{x}}=0$ is valid everywhere on $\mathbb{R}^2$. On the other hand, $\mathrm{e}^{\mathrm{i}\mathbf{p}_m\mathbf{x}}$ are linearly independent, and we obtain $\xi_m=0$ for all m. □

By corollary 3, perturbations of both the Rashba and Dresselhaus Hamiltonians by negative potentials from $L^p\cap L^1$, $p>1$, have infinitely many eigenvalues below the threshold of the continuous spectrum. Another important example where corollary 3 can be applied is the Hamiltonian with both Rashba and Dresselhaus terms:

$$H_{\mathrm{RD}}=\begin{pmatrix}p^2 & \alpha_R(p_y+\mathrm{i}p_x)-\alpha_D(p_x+\mathrm{i}p_y)\\ \alpha_R(p_y-\mathrm{i}p_x)-\alpha_D(p_x-\mathrm{i}p_y) & p^2\end{pmatrix},$$

which is used for describing the ballistic spin transport through a two-dimensional mesoscopic metal/semiconductor/metal double junction in the presence of spin–orbit interaction [5]. In this case $\kappa=-(|\alpha_R|^2+|\alpha_D|^2)/4$, and S contains exactly two points: $S=\{\mathbf{p}_0,-\mathbf{p}_0\}$, with

$$\mathbf{p}_0=\begin{cases}\frac{1}{2\sqrt{2}}(\alpha_R+\alpha_D,-\alpha_R-\alpha_D), & \text{if } \alpha_R\alpha_D>0,\\ \frac{1}{2\sqrt{2}}(\alpha_R-\alpha_D,\alpha_R-\alpha_D), & \text{if } \alpha_R\alpha_D<0.\end{cases}$$

In virtue of corollary 3, $H_{\mathrm{RD}}+V$ for negative V has at least two eigenvalues below κ.

We note that theorem 1 delivers some quantitative information on the eigenvalues. If $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_N$ are the solutions to $\det(\mathsf{V_a} - \mu \mathsf{G_a}) = 0$, where $\mathsf{V_a}$ (respectively, $\mathsf{G_a}$) is the matrix with the entries $\langle \Psi_m | V \Psi_n \rangle$ (respectively, $\langle \Psi_m | \Psi_n \rangle$), $1 \leqslant m, n \leqslant N$, then, for sufficient small a, the nth eigenvalue E_n of H $(1 \leqslant n \leqslant N)$ obeys the estimate $E_n \leqslant \mu_n < \kappa$.

It is worth noting that the class of perturbations for which the above machinery works contains singular perturbations supported on sets of zero Lebesgue measure [1], in particular, the Dirac δ-functions supported by curves. The latter class of 'potentials' has been used e.g. in [4] for studying the effect of spin–orbit interaction on bound states of electrons. Nevertheless, accurate demonstrations in this case require rather cumbersome purely technical details and are outside of the scope of the communication. We remark only that H_R or H_D perturbed by the Dirac δ-function supported by a circle has infinite number of eigenvalues below the threshold of the continuum spectrum. On the other hand, point perturbations of these Hamiltonians with one-point supports have exactly one bound state below the continuum.

Acknowledgments

We are very grateful to A V Chaplik and L A Chernozatonskii for discussion and valuable remarks. J B and V G were supported in part by the Sonderforschungsbereich 647 'Raum, Zeit, Materie' and the program of German–Russian cooperation of the Deutsche Forschungsgemeinschaft (Kz. 436 RUS 113/785). KP was supported by the research fellowship of the Deutsche Forschungsgemeinschaft (PA 1555).

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